

A geometry characteristic for Banach space with c^1 -norm

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ABSTRACT. Let E be a Banach space and $S(E) = \{e \in E : \|e\| = 1\}$. In this paper, a geometry characteristic for E is presented by using a geometrical construct of $S(E)$. That is, the norm of E is of c^1 in $E \setminus \{0\}$ if and only if $S(E)$ is a c^1 -submanifold of E with $\text{codim} S(E) = 1$. The theorem is very clear, however, its proof is non-trivial, which shows an intrinsic connection between the continuous differentiability of the norm $\|\cdot\|$ in $E \setminus \{0\}$ and differential structure of $S(E)$.

Keywords: Banach space, geometry, non-linear analysis, global analysis

2000 Mathematics Subject Classification 54Exx 46Txx, 58B20,

1 Introduction and preliminary

Let E be a Banach space and $S(E) = \{e \in E : \|e\| = 1\}$. First of all, we take the following example to illustrate our idea in origin. Let $\|(x, y)\|_1 = \max\{|x|, |y|\}$ and $\|(x, y)\| = \sqrt{x^2 + y^2}$ on \mathbb{R}^2 . $S(\mathbb{R}^2)$ in the norm $\|\cdot\|_1$ is the square with length $\sqrt{2}$ of diagonal line and center at 0, and in the norm $\|\cdot\|$ is the unit circle with center 0. In the second case, $S(\mathbb{R}^2)$ is a c^1 -curve, but is not in the first case. This difference of $S(\mathbb{R}^2)$ comes from that one of the norms is of c^1 in $\mathbb{R}^2 \setminus \{0\}$ but the other is not. However, in general, when E is a Banach space with c^1 -norm in $E \setminus \{0\}$, it is not known whether the geometry structure of $S(E)$ is a characteristic for the Banach space E with c^1 -norm in $E \setminus \{0\}$. In this paper, the following theorem is proved: the norm $\|\cdot\|$ of Banach space E is of c^1 in $E \setminus \{0\}$ if and only if $S(E)$ is a c^1 -submanifold of E with $\text{codim} S(E) = 1$. The proof is non-trivial but rather complex. Now let us recall some theorems and definitions in global analysis, which are needed in the sequel.

Definition 1.1 (*[Z],[AMR],[M1-2]*) Let M be a topological space. M is called a c^k -Banach manifold ($k \geq 1$) provided that there is an atlas $\{(U_\lambda, \varphi_\lambda, E_\lambda)\}_{\lambda \in \Lambda}$ such that

- (i) $\bigcup_{\lambda \in \Lambda} U_\lambda = M$.
- (ii) $\varphi_\lambda : U_\lambda \rightarrow \varphi_\lambda(U_\lambda) \subset E_\lambda$ is a homeomorphism where E_λ is a Banach space.
- (iii) If $U_\lambda \cap U_\mu \neq \emptyset$, then $\varphi_\lambda \circ \varphi_\mu^{-1} : E_\mu \rightarrow E_\lambda$ and $\varphi_\mu \circ \varphi_\lambda^{-1} : E_\lambda \rightarrow E_\mu$ are of c^k . The atlas $\{(U_\lambda, \varphi_\lambda, E_\lambda)\}_{\lambda \in \Lambda}$ is said to be a c^k differential structure of M .

Definition 1.2 ([Z], [AMR], [M1-2]) Let E be a Banach space. A subset S of E is called a c^k submanifold of E if and only if for each $x \in S$, there exists an admissible chart (U, φ, E_φ) of E with $x \in U$ such that the following hold:

- (i) The chart space E_φ contains a linear, closed subspace E_0 which splits E_φ .
- (ii) The chart image $\varphi(U \cap S)$ is an open set in E_0 .
- (iii) $\varphi : U \rightarrow \varphi(U) \subset E_\varphi$ is a c^k -diffeomorphism.

(Note that if E is only a c^k -Banach manifold M , then the condition (iii) is the same as in Definition 1.1. Here the single chart set $\{(E, I, E)\}$ is an atlas of E and so, is simplified.)

Let E, F be Banach spaces, and U an open set in E .

Definition 1.3 ([Z], [AMR], [M4]) Suppose that $f : x_0 \in U \subset E \rightarrow F$ is a c^k map, $k \geq 1$. x_0 is said to be a regular point of f provided that the Fréchet derivative $(Df)(x_0)$ is surjective and its null space $N((Df)(x_0))$ splits E .

Definition 1.4 ([Z], [AMR], [M4]) $y_0 \in F$ is said to be a regular value of f if and only if either the preimage $f^{-1}(y_0)$ is empty or consists of only regular points.

Theorem 1.1 ([Z], [AMR], [M4]) If $y_0 \in F$ is a regular value of f , then the preimage $S = f^{-1}(y_0)$ is a c^k -submanifold of E with $T_x S = N((Df)(x))$ for each $x \in S$.

Theorem 1.2 (Local normal form) ([Z]) If $f : U \subset E \rightarrow F$ is a c^k -map, $k \geq 1$ and $e_0 \in U$, then there exist a neighborhood U_0 at e_0 and c^k diffeomorphism $\varphi : U_0 \rightarrow \varphi(U_0)$ with $\varphi(e_0) = 0$ and $\varphi'(e_0) = I$ such that

$$f(e) = (Df)(e_0)\varphi(e) + f(e_0) \quad \forall e \in U_0.$$

Definition 1.5 ([Z], [ABR]) Let M be a topological space. Two charts (U, φ, E_φ) and (V, ψ, E_ψ) are called c^1 -compatible if and only if $U \cap V = \emptyset$, or $\varphi \circ \psi^{-1}$ from E_ψ into E_φ and $\psi \circ \varphi^{-1}$ from E_φ into E_ψ are c^1 .

Recall that a curve $v(t)$ on the unit sphere of E is called to be of c^1 provided so is $(\varphi \circ v)(t)$ for an admissible chart (U, φ, E_φ) satisfying the conditions (i)-(iii) in Definition 1.2; the equivalent class $[v]$ generated by the curve v consists of all c^1 -curves $u(t)$ satisfying that $u(0) = v(0) = e$ and $(\varphi \circ v)'(0) = (\varphi \circ u)'(0)$, which is independent of the choice of the chart (U, φ, E_φ) . Let $T_e S(E)$ denote all of these equivalent classes, and we call it the tangent space of $S(E)$ at $e \in S(E)$.

Remark 1.1 (i) $u(t) \in [v(t)]$ if and only if $u'(0) = v'(0)$ and $u(0) = v(0) \in S(E)$.

(ii) Let (U, φ, E_φ) be an admissible chart of E at e satisfying the conditions (i)-(iii) in Definition 1.2. Then

$$T_e S(E) = (D\varphi^{-1})(e)E_0.$$

(Since $[v(t)]$ is determined uniquely by $v'(0) \in E$, $T_e S(E)$ is topology isomorphic to a closed subspace of E , written as $T_e S(E)$ still. In addition,

$$(\varphi \circ v)'(0) = (D\varphi)(e)v'(0) \in E_0,$$

and so, $T_e S(E) = (D\varphi^{-1})(e)E_0$.) (For details, see [Z] and [ABR].)

2 Some important lemmas and theorems

In this section, the main results are as follows. A local normal form of the c^1 -norm $\|\cdot\|$ of Banach space E is given, which means that $\|e\| - \|e_0\|$ is locally c^1 -diffeomorphic to a linear functional in E^* , and its proof includes a technique on constructing the chart at each $e \in S(E)$ such that $S(E)$ becomes a c^1 -submanifold of E ; if $S(E)$ is a c^1 -submanifold of E , then $P_{[e_0+\Delta e]}^N \rightarrow P_{[e_0]}^{N_0}$ as $\Delta e \rightarrow 0$, where $P_{[e_0+\Delta e]}^N$ and $P_{[e_0]}^{N_0}$ are the projections corresponding to the decompositions, $E = N \oplus [e_0 + \Delta e]$ and $E = N_0 \oplus [e_0]$, respectively, $N = T_{e_0+\Delta e} S(E)$, $N_0 = T_{e_0} S(E)$, $[\cdot]$ denotes the one dimensional subspace generated by the vector in the bracket, and \oplus the topological direct sum. (This result is crucial to the proof of the theorem. However, the result itself seems to be very interesting and available for the study of infinite dimensional geometry.) In order to shorten the proof of the main theorem, we first provide its part conclusion and preliminary theorems, lemmas as preparations, some of which themselves are very interesting and useful.

Lemma 2.1 *If the norm $\|\cdot\|$ of Banach space E is Fréchet differentiable at the nonzero point e_0 . Then*

$$(D\|\cdot\|)(e_0)e_0 = \|e_0\|.$$

Proof. Let $\Delta e = \lambda e_0$. Then for λ small enough,

$$\|e + \Delta e\| - \|e_0\| = \lambda\|e_0\| = (D\|\cdot\|)(e_0)\lambda e_0 + o(\|\Delta e\|),$$

where the term $o(\|\Delta e\|)$ means $\lim_{\|\Delta e\| \rightarrow 0} \frac{o(\|\Delta e\|)}{\|\Delta e\|} = 0$. So

$$\|e_0\| = (D\|\cdot\|)(e_0)e_0 + \lim_{\lambda \rightarrow 0} \frac{o(\|\Delta e\|)}{\lambda} = (D\|\cdot\|)(e_0)e_0.$$

(note $\lim_{\lambda \rightarrow 0} \frac{o(\|\Delta e\|)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{\|e_0\|o(\|\Delta e\|)}{\|\Delta e\|} = 0$.)

□

Remark 2.1 Let $N((D\|\cdot\|)(e_0))$ denote the null space of $(D\|\cdot\|)(e_0)$, then, under the assumption of Lemma 2.1, $e_0 \notin N((D\|\cdot\|)(e_0))$ whenever $e_0 \neq 0$.

The following lemma is immediate from Theorem 1.2, however, it is important to establish the atlas of the spheres in Banach space with c^1 -norm.

Lemma 2.2 Suppose that the norm $\|\cdot\|$ of Banach space E is of c^1 in $E \setminus \{0\}$. Then for a nonzero $e_0 \in E$, there exist a neighborhood U_0 at e_0 and a diffeomorphism $\varphi : U_0 \rightarrow \varphi(U_0)$ with $\varphi(e_0) = 0$ and $\varphi'(e_0) = I$ such that

$$\|e\| = (D\|\cdot\|)(e_0)\varphi(e) + \|e_0\| \quad \forall e \in U_0.$$

Proof. By Remark 2.1, $(D\|\cdot\|)(e_0)e_0 \neq 0$ for each $e_0 \in E \setminus \{0\}$. Let $N = N((D\|\cdot\|)(e_0))$, then one has

$$E = N((D\|\cdot\|)(e_0)) \oplus [e_0] \quad \text{and} \quad I = P_N^{[e_0]} + P_{[e_0]}^N.$$

Let T^+ denote a right inverse of $(D\|\cdot\|)(e_0)$ such that

$$T^+(D\|\cdot\|)(e_0) = P_{[e_0]}^N \quad \text{and} \quad (D\|\cdot\|)(e_0)T^+r = r \quad \forall r \in \mathbb{R}. \quad (2.1)$$

(For details see [N].) Let

$$\varphi(e) = P_N^{[e_0]}e + T^+(\|e\| - \|e_0\|). \quad (2.2)$$

Evidently, $\varphi(e_0) = 0$ and

$$(D\varphi)(e_0) = P_N^{[e_0]} + T^+(D\|\cdot\|)(e_0) = P_N^{[e_0]} + P_{[e_0]}^N = I;$$

it is clear by (2.2) that φ is a diffeomorphism due to the Inverse Mapping Theorem, i.e., there is a neighborhood U_0 at e_0 such that $\varphi : U_0 \rightarrow \varphi(U_0)$ is a diffeomorphism. Note that $(D\|\cdot\|)(e_0)P_N^{[e_0]} = 0$ and $(D\|\cdot\|)(e_0)T^+(\|e\| - \|e_0\|) = \|e\| - \|e_0\|$ by (2.1). Consequently, we obtain

$$\|e\| = (D\|\cdot\|)(e_0)\varphi(e) + \|e_0\| \quad \forall e \in U_0.$$

□

Suppose that the Banach space E has the decomposition $E = E_0 \oplus E_1$. Let $e_0 = P_{E_0}^{E_1}e$, $e_1 = P_{E_1}^{E_0}e$, and $E_* = \{(e_0, e_1) : \forall e_0 \in E_0 \text{ and } e_1 \in E_1\}$. Define the norm $\|\cdot\|_*$ in E_* by $\|(e_0, e_1)\|_* = \max\{\|e_0\|, \|e_1\|\}$ for each $(e_0, e_1) \in E_*$. Evidently, $(E_*, \|\cdot\|_*)$ is a Banach space. The following lemma is convenient for mathematical calculus, as one will see in the next section, although it is simple.

Lemma 2.3 Let $\Gamma : B(E, \|\cdot\|) \rightarrow B(E_*, \|\cdot\|_*)$ be defined by $\Gamma(e) = (e_0, e_1) = (P_{E_0}^{E_1}e, P_{E_1}^{E_0}e)$ for any $e \in E$. Then $\Gamma \in B^\times((E, \|\cdot\|), (E_*, \|\cdot\|_*))$ and

$$\|(e_0, e_1)\|_* \leq \|\Gamma\|\|e\| \quad \text{and} \quad \|e\| \leq \|\Gamma^{-1}\|\|(e_0, e_1)\|_*,$$

where $B^\times((E, \|\cdot\|), (E_*, \|\cdot\|_*))$ is the set of all invertible operators in $B((E, \|\cdot\|), (E_*, \|\cdot\|_*))$. For abbreviation, write $B^\times(E, E_*)$ and $B(E, E_*)$ for them, respectively.

Proof. Since $N(\Gamma) = \{0\}$, $\Gamma^{-1}(e_0, e_1) = e_0 + e_1$ for any $(e_0, e_1) \in E_*$ and $\|\Gamma\| \leq \max\{\|P_{E_1}^{E_0}\| \cdot \|P_{E_0}^{E_1}\|\}$, the lemma is obvious from the Hahn-Banach Theorem. \square

Theorem 2.1 *Suppose that the norm $\|\cdot\|$ of Banach space E is of c^1 in $E \setminus \{0\}$. Then $S = S(E)$ is a c^1 -submanifold of E with $\text{codim} S = 1$.*

Proof. By Definition 2.1, the essential to proof of the theorem is to find an admissible chart of E at each $e_0 \in S$, fulfilling the conditions (i)-(iii) in Definition 1.2 and $\text{codim} E_0 = 1$. By Lemma 2.2, for each $e_0 \in S$, there exist a neighborhood U_0 at e_0 and c^1 diffeomorphism $\varphi : U_0 \rightarrow \varphi(U_0) \subset E$,

$$\varphi(e) = P_N^{[e_0]} e + T^+(\|e\| - \|e_0\|)$$

such that

$$\|e\| = (D\|\cdot\|)(e_0)\varphi(e) + \|e_0\| \quad \forall e \in U_0.$$

Hereby we see $\varphi(S \cap U_0) \subset \varphi(U_0) \cap N$. Conversely, for any $n \in \varphi(U_0) \cap N$, let $n = \varphi(n_*) \in N$ for some $n_* \in U_0$, then by the preceding equality, $\|n_*\| = \|e_0\| = 1$, and so $n_* \in \varphi(S \cap U_0)$. This shows $\varphi(S \cap U_0) (= \varphi(U_0) \cap N)$ is an open set in N . In addition, $\text{codim} N = \dim E/N = 1$ since $(D\|\cdot\|)(e_0)e_0 \neq 0$. We now conclude that (U_0, φ, E) is the required chart, $E_0 = N$ splits E and $\text{codim} E_0 = \dim(E/N) = 1$. \square

The following theorem is intuitive in geometry.

Theorem 2.2 *If $S = S(E) = \{e \in E : \|e\| = 1\}$ is a c^1 -submanifold of Banach space E , then $S_r = \{e \in E : \|e\| = r\}$, $r > 0$, is also a c^1 -submanifold of E and $T_{e_0}S = T_{e_1}S_r$ for any $e_1 \in S_r$, $e_1 = re_0$, i.e., the tangent hyperplane $T_{e_0}S + e_0$ of S at e_0 and $T_{e_1}S_r + e_1$ of S_r at e_1 mutual are parallel.*

Proof. Because of c^1 -submanifold S of E and by Definition 1.2, for any $e_0 \in S$, there exist an admissible chart (U, φ, E_φ) of E at e_0 and a closed subspace E_0 of E_φ such that $\text{codim} E_0 = \dim(E_\varphi/E_0) = 1$, $\varphi(U \cap S)$ is an open set in E_0 , and $\varphi : U \rightarrow \varphi(U)$ is an c^1 -diffeomorphism. Let $L_r e = re \forall e \in E$ for $r \neq 0$. Obviously, $L_r \in B^\times(E)$. So both $U_1 = rU$ and $r\varphi(U)$ are open sets in E . Let

$$\varphi_1(e) = (L_r \circ \varphi \circ L_r^{-1})(e) \quad \forall e \in E.$$

Clearly, $\varphi_1(U_1) : U_1 \rightarrow r\varphi(U)$ is a c^1 diffeomorphism, and $\varphi_1(S_r \cap U_1) = r\varphi(U \cap S)$ an open set in E_0 . Then by Definition 1.2, $(U_1, \varphi_1, E_\varphi)$ is an admissible of E at any $e_1 \in S_r$, which makes that S_r is a c^1 -submanifold of E . By Remark 1.1, $T_{e_0}S = (D\varphi^{-1})(e_0)E_0$ and $T_{e_1}S_r = (D\varphi_1^{-1})(e_1)E_0$. Evidently,

$$(D\varphi_1)(e_1) = L_r \cdot (D\varphi)(e_0) \cdot L_r^{-1},$$

from which it follows

$$\begin{aligned} T_{e_1}S_r &= (L_r \cdot (D\varphi^{-1})(e_0) \cdot L_r^{-1})E_0 = (L_r \cdot (D\varphi^{-1})(e_0))E_0 \\ &= r(D\varphi^{-1})(e_0)E_0 = T_{e_0}S. \end{aligned}$$

\square

Lemma 2.4 *Let $E = N_0 \oplus R_0$, then for any closed subspace R_1 of E satisfying $E = N_0 \oplus R_1$, there exists an operator $\alpha \in B(R_0, N_0)$ such that*

$$R_1 = \{e_0 + \alpha(e_0) : \forall e_0 \in R_0\}$$

and

$$P_{R_1}^{N_0} - P_{R_0}^{N_0} = \alpha \circ P_{R_0}^{N_0}.$$

Proof. Evidently,

$$P_{R_1}^{N_0} P_{R_0}^{N_0} e = P_{R_1}^{N_0} (P_{R_0}^{N_0} e + P_{N_0}^{R_0} e) = P_{R_1}^{N_0} e = e \quad \forall e \in R_1$$

and

$$P_{R_0}^{N_0} P_{R_1}^{N_0} e = P_{R_0}^{N_0} (P_{R_1}^{N_0} e + P_{N_0}^{R_1} e) = P_{R_0}^{N_0} e = e \quad \forall e \in R_0,$$

i.e., $P_{R_0}^{N_0} |_{R_1}$ is the inverse of $P_{R_1}^{N_0} |_{R_0}$. Let $e_0 = P_{R_0}^{N_0} e$ for $e \in R_1$. Then for each $e \in R_1$,

$$e = P_{R_0}^{N_0} e + P_{N_0}^{R_0} e = e_0 + P_{N_0}^{R_0} P_{R_1}^{N_0} e_0,$$

so $\alpha = P_{N_0}^{R_0} P_{R_1}^{N_0} |_{R_0} \in B(R_0, N_0)$ such that $R_1 = \{e_0 + \alpha(e_0) : \forall e_0 \in R_0\}$. Hereby,

$$P_{R_1}^{N_0} e = P_{R_0}^{N_0} e + \alpha(P_{R_0}^{N_0} e) = (I + \alpha)P_{R_0}^{N_0} e \quad \forall e \in E,$$

i.e.,

$$P_{R_1}^{N_0} - P_{R_0}^{N_0} = \alpha \circ P_{R_0}^{N_0}.$$

□

The next theorem shows some interesting geometrical significance, which is available for the study of infinite dimensional geometry. It is also necessary for the proof of the main theorem below.

Theorem 2.3 *Suppose that $S = S(E)$ is a c^1 -submanifold of E with $\text{codim} S = 1$. Let N_0, N be the tangent spaces of S_{r_0} at e_0 and S_r at $e_0 + \Delta e$, respectively, where $r_0 = \|e_0\|$ and $r = \|e_0 + \Delta e\|$. Then*

$$P_{[e_0 + \Delta e]}^N \rightarrow P_{[e_0]}^{N_0} \quad \text{as } \Delta e \rightarrow 0.$$

Proof. Let (U, φ, E_φ) at the point $e_0 \in E \setminus \{0\}$ be an admissible chart of E satisfying the conditions (i)-(iii) in Definition 1.2. Assume that $\|\Delta e\|$ is small enough such that $e_0 + \Delta e \in U$. Then $(D\varphi^{-1})(e) \in B^\times(E_\varphi, E)$ for e near e_0 fulfils

$$N_0 = T_{e_0} S_{r_0} = (D\varphi^{-1})(e_0) E_0 \quad \text{and} \quad N = T_{e_0 + \Delta e} S_r = (D\varphi^{-1})(e_0 + \Delta e) E_0.$$

Hereby

$$N = (D\varphi^{-1})(e_0 + \Delta e)(D\varphi)(e_0) N_0.$$

Because of $\text{codim} S_{r_0} = \text{codim} S_r = 1$ one can conclude

$$E = N_0 \oplus [e_0] = N \oplus [e_0 + \Delta e].$$

Let $e_* = (D\varphi^{-1})(e_0 + \Delta e)(D\varphi)(e_0)e_0$ and $\Psi = (D\psi^{-1})(e_0 + \Delta e)(D\varphi)(e_0)$. It then follows

$$E = N \oplus [e_*] \quad \text{and} \quad P_{[e_*]}^N = \Psi P_{[e_0]}^{N_0} \Psi^{-1}.$$

By the continuity of Ψ and Ψ^{-1} one can assert

$$P_{[e_*]}^N \rightarrow P_{[e_0]}^{N_0} \quad \text{as} \quad \Delta e \rightarrow 0. \quad (2.3)$$

We now are in the position to prove the theorem. By (2.3), we have

$$P_{[e_*]}^N(e_0 + \Delta e) = P_{[e_*]}^N e_0 + P_{[e_*]}^N \Delta e \rightarrow P_{[e_0]}^{N_0} e_0 = e_0 \quad \text{as} \quad \Delta e \rightarrow 0.$$

By the determination of e_* and the continuity of Ψ it is immediate that $e_* \rightarrow e_0$ as $\Delta e \rightarrow 0$. While by Lemma 2.4, there is an operator $\alpha \in B([e_*], N)$ such that $[e_0 + \Delta e] = \{\lambda e_* + \lambda \alpha(e_*) : \forall \lambda \in \mathbb{R}\}$ so that $P_{[e_*]}^N(e_0 + \Delta e) = \lambda e_*$. Then $e_0 = (\lim_{\Delta e \rightarrow 0} \lambda) e_0$ and so, $\lim_{\Delta e \rightarrow 0} \lambda = 1$. We next go to show $\lim_{\Delta e \rightarrow 0} \alpha(e_*) = 0$. Obviously, (2.3) implies $P_N^{[e_*]} \rightarrow P_{N_0}^{[e_0]}$ as $\Delta e \rightarrow 0$. So

$$\|P_N^{[e_*]} \Delta e\| \leq \|P_N^{[e_*]}\| \|\Delta e\| \rightarrow 0 \quad \text{and} \quad P_N^{[e_*]} e_0 \rightarrow P_{N_0}^{[e_0]} e_0 = 0$$

as $\|\Delta e\| \rightarrow 0$. Hence it follows $\alpha(e_*) \rightarrow 0$ as $\Delta e \rightarrow 0$ from $P_N^{[e_*]}(e_0 + \Delta e) = P_N^{[e_*]} e_0 + P_N^{[e_*]} \Delta e = \lambda \alpha(e_*)$ and $\lambda \rightarrow 1$. In order to complete the proof, we also need to show

$$\|P_{[e_0 + \Delta e]}^N - P_{[e_*]}^N\| \rightarrow 0 \quad \text{as} \quad \Delta e \rightarrow 0.$$

Let $P_N^{[e_*]} h = \lambda e_*$ for any $h \in E$. Clearly, $|\lambda| = \frac{\|P_N^{[e_*]} h\|}{\|e_*\|}$. Then

$$\|\alpha(P_N^{[e_*]} h)\| = |\lambda| \|\alpha(e_*)\| = \frac{\|\alpha(e_*)\|}{\|e_*\|} \|P_N^{[e_*]} h\|,$$

so

$$\|\alpha\| \leq \frac{\|\alpha(e_*)\|}{\|e_*\|} \|P_N^{[e_*]}\|.$$

In addition, by Lemma 2.4,

$$P_{[e_0 + \Delta e]}^N - P_{[e_*]}^N = \alpha \circ P_{[e_*]}^N.$$

Thus, since $\|P_{[e_*]}^N\| \rightarrow \|P_{[e_0]}^{N_0}\|$, $\|\alpha(e_*)\| \rightarrow 0$, and $\|e_*\| \rightarrow \|e_0\|$, one can assert

$$\|P_{[e_0 + \Delta e]}^N - P_{[e_*]}^N\| \rightarrow 0 \quad \text{as} \quad \Delta e \rightarrow 0.$$

Finally, from

$$P_{[e_0 + \Delta e]}^N - P_{[e_0]}^{N_0} = P_{[e_0 + \Delta e]}^N - P_{[e_*]}^N + P_{[e_*]}^N - P_{[e_0]}^{N_0}$$

it follows

$$P_{[e_0 + \Delta e]}^N \rightarrow P_{[e_0]}^{N_0} \quad \text{as} \quad \Delta e \rightarrow 0.$$

□

3 Main result

Theorem 3.1 *Let E be a Banach space. If $S = S(E)$ is a c^1 -submanifold of E with $\text{codim} S = 1$, then the norm $\|\cdot\|$ of E is of c^1 in $E \setminus \{0\}$.*

Proof. By Definition 1.2, since S is a c^1 -submanifold of E , with $\text{codim} S = 1$, one has that for each $e_0 \in S$, there exists a c^1 admissible chart (U, φ, E_φ) of E at e_0 such that $E_0 \subset E_\varphi$ splits E_φ , $\varphi(U \cap S)$ is an open set in E_0 , and

$$\varphi : U \rightarrow \varphi(U) \quad \text{and} \quad \varphi^{-1} : \varphi(U) \rightarrow U$$

are both c^1 -homeomorphisms. Let $\varphi(e_0) = e_\varphi^0 \in E_0$. Then there exists a positive number η such that

$$\varphi^{-1}(e_\varphi^0 + \Delta e_\varphi) - \varphi^{-1}(e_\varphi^0) = (D\varphi^{-1})(e_\varphi^0)\Delta e_\varphi + o(\|\Delta e_\varphi\|) \quad (3.1)$$

and

$$e_\varphi^0 + \Delta e_\varphi \in \varphi(U \cap S). \quad (3.2)$$

whenever $\Delta e_\varphi \in E_0$ such that $\|\Delta e_\varphi\| < \eta$. Let $\tau = (D\varphi^{-1})(e_\varphi^0)\Delta e_\varphi \in T_{e_0}S$ (see Remark 1.1). It is obvious that $\|\tau\| < \frac{\eta}{\|(D\varphi)(e_0)\|}$ implies $\|\Delta e_\varphi\| < \eta$ since $\|\Delta e_\varphi\| = \|(D\varphi)(e_0)\tau\| \leq \|(D\varphi)(e_0)\|\|\tau\|$. Thus it follows from (3.1) and (3.2)

$$\|e_0 + \tau + o(\|\tau\|)\| - \|e_0\| = 0 \quad \text{whenever} \quad \|\tau\| < \frac{\eta}{\|(D\varphi)(e_0)\|}, \quad (3.3)$$

(note $\|\varphi^{-1}(e_\varphi^0 + \Delta e_\varphi)\| = \|e_0\|$ by (3.2)). Moreover, by (3.3) and the triangular inequality for the norm $\|\cdot\|$, it is easy to examine

$$\|e_0 + \tau\| - \|e_0\| \geq -o(\|\tau\|) \quad \text{and} \quad o(\|\tau\|) \geq \|e_0 + \tau\| - \|e_0\|,$$

i.e., $\|e_0 + \tau\| - \|e_0\|$ is a higher order infinitesimal than $\|\tau\|$. Hereby one gets

$$\|e_0 + \tau\| - \|e_0\| = o(\|\tau\|) \quad (3.4)$$

whenever $\|\tau\| < \frac{\eta}{\|(D\varphi)(e_0)\|}$. Hereafter, $o(\|\tau\|)$ is a real number. We claim that e_0 is not in $T_{e_0}S$, since otherwise it leads to the contradiction that by (3.4)

$$\|e_0 + \lambda e_0\| - \|e_0\| = o(\|\tau\|) \quad \text{whenever} \quad |\lambda| < \frac{\eta}{\|(D\varphi)(e_0)\|}$$

but by computing directly,

$$\|e_0 + \tau\| - \|e_0\| = \lambda\|e_0\|.$$

So, by $\text{codim} S = 1$, one has $E = N_0 \oplus [e_0]$ where $N_0 = T_{e_0}S$. Next we show that the norm $\|\cdot\|$ of E is Fréchet differentiable at each $e_0 \in S$. Let $h = \tau + \lambda e_0$ for any $h \in E$ where $\tau \in T_{e_0}S$. By computing directly,

$$\begin{aligned} & \|e_0 + (\tau + \lambda e_0)\| - \|e_0\| \\ &= \|e_0 + \tau + \lambda e_0\| - \|e_0 + \lambda e_0\| + \|e_0 + \lambda e_0\| - \|e_0\| \\ &= (1 + \lambda)\|e_0 + \frac{\tau}{1 + \lambda}\| - \|(1 + \lambda)e_0\| + \|(1 + \lambda)e_0\| - \|e_0\| \\ &= (1 + \lambda)\left\{\|e_0 + \frac{\tau}{1 + \lambda}\| - \|e_0\|\right\} + \lambda\|e_0\| \end{aligned}$$

for $|\lambda| < 1$. Further, applying (3.4) to $\|e_0 + \frac{\tau}{1+\lambda}\|$,

$$\|e_0 + (\tau + \lambda e_0)\| - \|e_0\| = \lambda\|e_0\| + (1 + \lambda)o(\|\frac{\tau}{1+\lambda}\|)$$

for $\|\frac{\tau}{1+\lambda}\| < \frac{\eta}{\|(D\varphi)(e_0)\|}$ and $|\lambda| < 1$. Since for $|\lambda| < 1$,

$$\frac{\|\tau\|}{1+\lambda} \rightarrow 0 \Leftrightarrow \|\tau\| \rightarrow 0$$

and

$$\frac{(1 + \lambda)o(\|\frac{\tau}{1+\lambda}\|)}{\|\tau\|} = \frac{o(\|\frac{\tau}{1+\lambda}\|)}{\frac{\|\tau\|}{1+\lambda}},$$

it is clear that $(1 + \lambda)o(\|\frac{\tau}{1+\lambda}\|)$, written still by $o(\|\tau\|)$, is also a higher order infinitesimal than $\|\tau\|$. Therefore, one can assert

$$\|e_0 + h\| - \|e_0\| = \lambda\|e_0\| + o(\|\tau\|), \quad (3.5)$$

whenever $\|\tau\| < \frac{\eta}{2\|(D\varphi)(e_0)\|}$ and $|\lambda| < \frac{1}{2}$. Let $\delta = \min\{\frac{\eta}{2\|(D\varphi)(e_0)\|}, \frac{1}{2}\}$, and $h = \tau + \lambda e_0$ for each $h \in E$, where $\tau \in T_{e_0}S$. Define $\Gamma h = (\tau, \lambda)$ by the same way as in Lemma 2.3. Then from

$$\|\Gamma\|\|h\| \geq \|\Gamma h\|_* = \max\{\|\tau\|, |\lambda|\}$$

it follows that for any h such that $\|h\| < \|\Gamma\|^{-1}\delta$,

$$\|\tau\| < \frac{\eta}{2\|(D\varphi)(e_0)\|} \quad \text{and} \quad |\lambda| < \frac{1}{2}.$$

Thus, by (3.4) we have

$$\|e_0 + h\| - \|e_0\| = \lambda\|e_0\| + o(\|\tau\|) \quad \text{whenever} \quad \|h\| < \|\Gamma\|^{-1}\delta. \quad (3.6)$$

In order to prove the Fréchet differentiability of the norm $\|\cdot\|$ in $T_{e_0}S$, we also have to show

$$\lim_{h \rightarrow 0} \frac{o(\|\tau\|)}{\|h\|} = \lim_{\tau \rightarrow 0} \frac{o(\|\tau\|)}{\|\tau\|}.$$

By Lemma 2.3 it is easy to see

$$\|h\| \rightarrow 0 \Leftrightarrow \|(\tau, \lambda)\|_* \rightarrow 0 \Rightarrow \|\tau\| \rightarrow 0,$$

and

$$\frac{o(\|\tau\|)}{\|h\|} \leq \frac{o(\|\tau\|)}{\|\Gamma\|^{-1}\|(\tau, \lambda)\|_*} \leq \frac{o(\|\tau\|)}{\|\Gamma\|^{-1}\|\tau\|}.$$

Then by (3.5)

$$\|e_0 + h\| - \|e_0\| = \lambda\|e_0\| + o(\|h\|) \quad \text{whenever} \quad \|h\| < \|\Gamma\|^{-1}\delta.$$

Let $h = P_{N_0}^{[e_0]}h + P_{[e_0]}^{N_0}h$, and $P_{[e_0]}^N h = \lambda e_0$ for each point $e_0 \in S$. Define a bounded linear functional $f_{e_0} \in E^*$ as follows:

$$f_{e_0}(h) = \lambda \|e_0\| \quad \forall h \in E,$$

which satisfies

$$|f_{e_0}(h)| = \|P_{[e_0]}^N h\| \leq \|P_{[e_0]}^N\| \|h\|.$$

Finally one gets by (3.6)

$$\|e_0 + h\| - \|e_0\| = f_{e_0}(h) + o(\|h\|) \quad \text{whenever } \|h\| < \|\Gamma\|^{-1}\delta.$$

This proves that

$$(D\|\cdot\|)(e_0)h = f_{e_0}(h),$$

i.e., the norm $\|\cdot\|$ is Fréchet differentiable at each $e_0 \in S$.

Next we show that the norm $\|\cdot\|$ is Fréchet differentiable for each $e \in E \setminus \{0\}$. Let $e_1 = \|e_1\|e_0$ for each $e_1 \in E \setminus \{0\}$, then $\|e_0\| = 1$. Replace e_0, φ, S and U above by $e_1, \varphi_1 = r\varphi, S_r$, and $U_1 = rU$, respectively, where $r = \|e_1\|$. Note that E_φ and E_0 keep invariant as shown in the proof of Theorem 2.2. Repeat the process above. Then, the following results follow in turn,

(i) there is a positive number η such that

$$\varphi_1^{-1}(e_{\varphi_1}^1 + \Delta e_\varphi) - \varphi_1^{-1}(e_{\varphi_1}^1) = (D\varphi_1^{-1})(e_{\varphi_1}^1)\Delta e_\varphi + o(\|\Delta e_\varphi\|)$$

whenever $\|\Delta e_\varphi\| < \eta$ and $\Delta e_\varphi \in E_0$.

(ii) let $h = \tau + \lambda e_1$ for any $h \in E$, where $\tau \in T_{e_1}S_r$, $\delta = \min\{\frac{\eta}{2(D\varphi_1)}, \frac{\|e_1\|}{2}\}$, and $\Gamma(h) = (\tau, \lambda e_1)$, then

$$\|e_1 + h\| - \|e_1\| = \lambda \|e_1\| + o(\|h\|) \quad \text{whenever } \|h\| < \|\Gamma\|^{-1}\delta.$$

(iii) for any $h \in E$, let $h = P_N^{[e_1]}h + P_{[e_1]}^N h$, and $P_{[e_1]}^N h = \lambda e_1$ (where $N = T_{e_0}S = T_{e_1}S_r$ by Theorem 2.2), then

$$(D\|\cdot\|)(e_1)h = f_{e_1}(h),$$

where f_{e_1} is the bounded linear functional determined by $f_{e_1}(h) = \lambda \|e_1\|$ as $P_{[e_1]}^N h = \lambda e_1$.

To the end of the proof, it remains to examine the continuity of $(D\|\cdot\|)$. Let (U, φ, E_φ) at any point $e_0 \in E \setminus \{0\}$ be an admissible chart of E satisfying the conditions (i)-(iii) in Definition 1.2. Assume that $\|\Delta e\|$ is small enough such that $e_0 + \Delta e \in U$. Let $r_0 = \|e_0\|$, $r = \|e_0 + \Delta e\|$. Thus

$$N_0 = T_{e_0}S_{r_0} = (D\varphi^{-1})(e_0)E_0 \quad \text{and} \quad N = T_{e_0+\Delta e}S_r = (D\varphi^{-1})(e_0 + \Delta e)E_0.$$

Hereby

$$N = (D\varphi^{-1})(e_0 + \Delta e)(D\varphi)(e_0)N_0.$$

Because of $\text{codim}S_{r_0} = \text{codim}S_r = 1$ one can conclude

$$E = N_0 \oplus [e_0] = N \oplus [e_0 + \Delta e].$$

By Theorem 2.3,

$$P_{[e_0 + \Delta e]}^N \rightarrow P_{[e_0]}^{N_0} \quad \text{as } \Delta e \rightarrow 0.$$

In addition,

$$P_{[e_0 + \Delta e]}^N h = \frac{1}{\|e_0 + \Delta e\|} f_{e_0 + \Delta e}(h)(e_0 + \Delta e) \quad P_{[e_0]}^{N_0} h = \frac{1}{\|e_0\|} f_{e_0}(h)e_0 \quad \forall h \in E.$$

Obviously, $e_0 + \Delta e \rightarrow e_0$ as $\Delta e \rightarrow 0$. Therefore, one asserts

$$f_{e_0 + \Delta e} \rightarrow f_{e_0} \quad \text{as } \Delta e \rightarrow 0.$$

Finally, one gets

$$(D\|\cdot\|)(e_0 + \Delta e) = \|e_0 + \Delta e\| f_{e_0 + \Delta e} \rightarrow (D\|\cdot\|)(e_0) = \|e_0\| f_{e_0} \quad \text{as } \Delta e \rightarrow 0.$$

i.e., $(D\|\cdot\|)(e)$ is continuous at each $e_0 \in E \setminus \{0\}$. \square

Combining Theorems 2.1 and 3.1 bears the main theorem in the paper:

Theorem 3.2 *Suppose that E is a Banach space. Then the norm $\|\cdot\|$ of E is of c^1 in $E \setminus \{0\}$ if and only if $S(E)$ is a c^1 -submanifold of E with $\text{codim}S(E) = 1$.*

Corollary 3.1 *Suppose that S is a c^1 submanifold of E . Let $N = T_{e_0}S$, $E = T_{e_0}S \oplus [e_0]$ for $e_0 \in S$, and $e = P_N^{[e_0]}e + \lambda e_0$. then $f_{e_0}(e) = \lambda \|e_0\| \in E^*$ fulfills*

$$(D\|\cdot\|)(e_0)h = f_{e_0}(h) \quad \forall h \in E.$$

4 Examples

The next two examples are interesting, which shows how to determinate the Fréchet differential of the norm $\|\cdot\|$ by geometrical knowledge, although they are simple.

Example 1 *Let $\|(x, y)\| = \sqrt{x^2 + y^2}$ for any $(x, y) \in \mathbb{R}^2$, and S be the unit circle with center 0. It is clear that the tangent line at a point $(x_0, y_0) \in S$ is the line perpendicular to the radial vector (x_0, y_0) , so that $N_0 = T_{(x_0, y_0)}S = \{(x, y) : xx_0 + yy_0 = 0\}$. Hence*

$$\mathbb{R}^2 = N_0 \oplus [(x_0, y_0)],$$

and

$$P_{[(x_0, y_0)]}^{N_0} h = \lambda(x_0, y_0) = \lambda \|(x_0, y_0)\| e_0$$

for any $h \in \mathbb{R}^2$, where $e_0 = \frac{(x_0, y_0)}{\sqrt{x_0^2 + y_0^2}}$. As is well-known from element geometry, reads the formula of the distance from h to the tangent line of S at (x_0, y_0)

$$f_{(x_0, y_0)}h = \lambda \|(x_0, y_0)\| = \frac{x_0 h_1 + y_0 h_2}{\sqrt{x_0^2 + y_0^2}} = x_0 h_1 + y_0 h_2,$$

where $h = (h_1, h_2)$. By Theorem 3.1,

$$D(\sqrt{x^2 + y^2})(x_0, y_0)h = x_0 h_1 + y_0 h_2.$$

Example 2 Let H be a Hilbert Space, \langle, \rangle denote its inner product, and $\|h\| = \sqrt{\langle h, h \rangle}$. Let S be the unit sphere in H and $h_0 \in S$. Then the subspace N_0 perpendicular to h_0 is just $T_{h_0}S$ and $N_0 = T_{h_0}S = \{h \in H : \langle h_0, h \rangle = 0\}$. Since $\text{codim}S = 1$

$$H = N_0 \oplus [h_0].$$

Evidently,

$$P_{[h_0]}^{N_0}h = \lambda h_0 = \lambda \|h_0\| e_0 = \langle h, h_0 \rangle e_0 \quad \forall h \in H,$$

where $e_0 = \frac{h_0}{\|h_0\|}$. By Theorem 3.1,

$$(D\sqrt{\langle h, h \rangle})(h_0)\Delta h = \langle h_0, \Delta h \rangle \quad \forall \Delta h \in H.$$

Acknowledgement I would like to thank Professor Yuwen Wang for telling me that he has considered the relationship between smooth Banach space and smooth unit sphere for several years. This work is supported by the National Science Foundation of China (Grant No. 10671049 and 10771101).

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